A Neglected Thomistic Text on the Foundation of Mathematics

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EVER since the time of Plato philosophers have been concerned with the nature of mathematics and its relation to reality. The discovery in modern times of new types of mathematics, such as the non-Euclidean geometries, has added new difficulties to the problem, while making its solution more urgent. The ancient view of mathematics as the science of quantity, or of space and number, appears to the modern mind as narrow, in the light of such discoveries as projective geometry and group theory. We are told that "Geometry, inasmuch as it is concerned with real space, is no longer considered a part of pure mathematics; like mechanics and physics, it belongs among the applications of mathematics."[1] This is in agreement with the views of Kant, who set the tone for the modern philosophy of mathematics when he declared that mathematical knowledge is that which reason gains not from concepts but from the *construction* of concepts.[2] No less an authority in mathematical physics than Albert Einstein asserted, in the Kantian spirit, that mathematics is a product of the human mind independent of all experience.[3] Along with this emancipation of mathematics from reality we find the obliteration, in some circles, of the distinction between mathematics and logic, under the influence of the general arithmetic of hypercomplex numbers, axiomatic investigations, set theory and symbolic logic.[4]

The observations of St. Thomas on mathematics take us back to an era when mathematical knowledge was still in its infancy. His knowledge of mathematics was limited to Euclidean geometry and arithmetic. An Arabian treatise on algebra was translated into Latin in the twelfth century by Robert of Chester, but St. Thomas shows no knowledge of it.[5] As for his philosophical views on mathematics, his interpreters are far from being in agreement. Frequently his conception of the object of mathematics is understood to be

[1] H. Weyl, *Philosophy of Mathematics and Natural Science* (Princeton 1949) p. 62.

[2] I. Kant, *Critique of Pure Reason*, trans. N. K. Smith (London 1950) p. 577. Cited by H. Weyl, *op. cit.*, p. 65.

[3] A. Einstein, *Geometrie und Erfahrung* (Berlin, 1921). Cited by P. Hoenen, 'De Philosophia Scholastica Cognitionis Geometricae'. *Gregorianum* XIX (1938) 505.

[4] Cf. H. Weyl, *op. cit.*, p. 62.

[5] L. Karpinski. *Robert of Chester's Latin Translation of the Algebra of Al-Khowarizmi* (New York, 1915).

frankly, and even naively realistic. For example, J. Gredt tells us that, according to Aristotle and St. Thomas, the object of mathematics is real quantity. The mathematician abstracts the essence of quantity, while leaving out of consideration its relation to the real being of corporeal substance in which it exists. Quantity, thus considered, is not a being of reason (*ens rationis*), but a real being (*ens reale*). Of course, this is true only for Euclidean geometry and arithmetic. Modern mathematicians, Fr. Gredt continues, extend mathematical speculation to fictitious quantity, which is not a real being but only a being of reason. This is a special, transcendental mathematics, essentially distinct from real mathematics.[6]

More recently, V. Smith has stated without qualification that for St. Thomas geometry is "not a study of an ideal order but a science of the real world."[7] Its object is intelligible matter, or in other words substance with unterminated dimensions of quantity, which is something "truly real." He concludes: "Euclidean geometry is the science of what is real but not physical, imaginable but not sensible, truly essential but not natural and mobile." It is accordingly a science in its own right with an object existing in the physical world.[8]

Other Thomists, while conceding that mathematical concepts have a real foundation, stress the role played by the intellect in their formation. In his monumental *Les degrés du savoir* J. Maritain agrees with Fr. Gredt in characterizing the objects of Euclidean geometry and the arithmetic of whole numbers as *entia realia*, in distinction to the objects of modern types of mathematics which are *entia rationis*.[9] The basis of this distinction is the fact that the former can exist outside the mind in the material world, whereas the latter cannot. Straight lines, circles, and whole numbers are found in sensible things, but not irrational numbers or the constructions of non-Euclidean geometry. Maritain emphasizes, however, that mathematical entities found in the real world acquire an ideal purity in the mind of the mathematician which they do not have in their real existence. Through the abstractive act-

[6] "Obiectum Matheseos est quantitas realis ita tamen secundum quidditatem suam abstracte et inadaequate considerata, ut non dicat ordinem ad esse reale in substantia corporea seu in ente mobili... Quantitas ita considerata non est quidem ens rationis, sed ens reale, tamen ita abstracte consideratur, ut abstrahat etiam ab esse reali et esse rationis. Recentes mathematici speculationem mathematicam usque ad quantitatem fictam extendunt, quae non est ens reale, sed rationis tantum, ut est quarta dimensio, quae secundum essentiam suam positive excludit ordinem ad esse reale. Ita constituitur Mathesis quaedam specialis, quae vocatur Mathesis transcendentalis et quae a Mathesi reali essentialiter distinguitur neque ad eam pertinet nisi reductive." J. Gredt, *Elementa Philosophiae Aristotelico-Thomisticae* (Freiburg 1929) vol. I, p. 194.

[7] V. Smith, *St. Thomas on the Object of Geometry* (Milwaukee 1954) p. 65.

[8] V. Smith, *op. cit.*, pp. 84, 85.

[9] J. Maritain, *Distinguer pour unir, ou Les degrés du savoir* (Paris 1932) pp. 283-285.

ivity of the intellect these entities undergo an ideal purification which affects not only their mode of being but their very definition. There are no points, lines or whole numbers in the real world with the conditions proper to mathematical abstraction: in nature there are no points without lines, lines without thickness, or abstract numbers.[10]

According to John of St. Thomas, mathematical quantity must be distinguished both from imaginary quantity, which is an *ens rationis*, and from real quantity. The quantity considered by the mathematician is not precisely a being of reason nor a real being but is indifferent to both. This accounts for the fact that mathematical demonstrations are equally valid for both real and imaginary quantity.[11]

It would be presumptuous in this brief note to pass judgment on the accuracy of these different interpretations of the thought of St. Thomas. Our purpose is simply to call attention to a text of St. Thomas on the foundation of mathematics which, to the present writer's knowledge, has not been taken into account in estimating his views on mathematics. Although brief, the text is significant for it places mathematical notions on the same level as those of logic, as far as their foundation in reality is concerned. The immediate foundation for both is said to be the activity of the intellect; only remotely do they have a basis in reality.

The text in question is found in St. Thomas' *Scriptum super libros Sententiarum*, I, d. 2, a. 3. Fr. A. Dondaine has shown that this article was a separate *Quaestio Disputata* composed by St. Thomas at Rome between the years 1265 and 1267, and inserted by the author himself in his commentary on the *Sentences*.[12] Unlike the rest of the commentary, therefore, this article is not an early writing of St. Thomas but dates from his mature years.

The subject of the article is the distinction between the divine attributes and their foundation in God. St. Thomas contends that divine attributes, such as wisdom and goodness, are entirely one in God Himself, but they differ in *ratio*. By *ratio* he means "that which the intellect apprehends about the

[10] J. Maritain, *op. cit.*, pp. 327, 328.

[11] John of St. Thomas, *Cursus theologicus*, I, disp. 6, a. 2 (Paris 1931) vol. I, p. 534. For P. Hoenen, the objects of geometry are not a pure creation, but they presuppose a constructive activity of our mind exercised upon a given matter, namely extension. Geometry is a discovery of extension, with its properties, in the physical, concrete object, and this physical extension must have geometrical properties. See P. Hoenen, "Pour une philosophie de la connaissance de l'étendu physique," *Gregorianum* XXX (1949) 195, 196. For E. Maziarz, mathematical beings are *entia realia* and not merely *entia rationis* (which he identifies with logical beings). They are not purely fictitious, yet they are as such incapable of extra-mental existence. See his *The Philosophy of Mathematics* (New York 1950) p. 208, and p. 227, note 125.

[12] Cf. A. Dondaine, 'Saint Thomas et la dispute des attributs divins (I Sent., d. 2, a. 3) authenticité et origine,' *Archivum Fratrum Praedicatorum* VIII (1938) 253-62.

meaning of a name." If the object of the intellect can be defined, its *ratio* is identical with its definition. St. Thomas warns us that *ratio* is not to be confused with the concept existing in the intellect; it is rather the "intention" of a concept (*sed significat intentionem hujus conceptionis*). In short, the *ratio* is the meaning or significance of a concept. Now is there any sense in which a *ratio* so understood exists in reality? Yes, St. Thomas replies. Of course the *ratio* itself has no real existence, any more than the concept does to which the *ratio* is attached. But a *ratio* may be said to exist in reality if there is something real which corresponds to the concept, as an object signified corresponds to its sign.[13]

In order to clarify this latter point, St. Thomas describes three ways in which concepts are related to reality. 1) Some concepts are likenesses of realities existing outside the soul; for example the concept of man. A concept of this sort has an immediate foundation in reality, so that the truth of the concept is caused by reality itself and the name signifying the concept is properly predicated of reality. 2) Some concepts are not likenesses of realities existing outside the soul, but the intellect comes upon (*adinvenit*) their intentions as a consequence of the way it understands reality. For example, the concept "genus" is not the likeness of a reality outside the soul; but from the fact that the intellect understands that there are a number of species of animals it attributes to animal the intention of "genus". An intention of this sort has only a remote foundation in the real world; its proximate foundation is in the intellect itself. The same is true, St. Thomas continues, of all other intentions which follow upon our way of understanding, for example mathematical abstractions (*abstractio mathematicorum*), and the like. 3) Some con-

[13] "Quantum ad primum pertinet, sciendum est, quod ratio, prout hic sumitur, nihil aliud est quam id quod apprehendit intellectus de significatione alicujus nominis: et hoc in his quae habent definitionem est ipsa rei definitio, secundum quod philosophus dicit, IV *Metaph.*, text. 11: "Ratio quam significat nomen est definitio." Sed quaedam dicuntur habere rationem sic dictam, quae non definiuntur, sicut quantitas et qualitas, et hujusmodi, quae non definiuntur, quia sunt genera generalissima. Et tamen ratio qualitatis est id quod significatur nomine qualitatis; et hoc est illud ex quo qualitas habet quod sit qualitas. Unde non refert, utrum illa quae dicuntur habere rationem, habeant vel non habeant definitionem. Et sic patet quod ratio sapientiae quae de Deo dicitur, est id quod concipitur de significatione hujus nominis, quamvis ipsa sapientia divina definiri non possit. Nec tamen hoc nomen 'ratio' significat ipsam conceptionem, quia hoc significatur per nomen rei; sed significat intentionem hujus conceptionis, sicut et hoc nomen 'definitio', et alia nomina secundae impositionis.

"Et ex hoc patet secundum, scilicet qualiter ratio dicatur esse in re. Non enim hoc dicitur, quasi ipsa intentio quam significat nomen rationis, sit in re; aut etiam ipsa conceptio, cui convenit talis intentio, sit in re extra animam, cum sit in anima sicut in subjecto: sed dicitur esse in re, inquantum in re extra animam est aliquid quod respondet conceptioni animae, sicut significatum signo." *Sent.*, I, d. 2, a. 3; ed. P. Mandonnet (Paris 1929) vol. I, pp. 66, 67.

cepts have no foundation in reality, either remote or proximate, as in the case of the concept of a chimera. This is not the concept of a reality, nor does it follow upon the way we understand reality. For this reason St. Thomas calls it a false concept.

St. Thomas concludes that, properly speaking, a *ratio* can be said to exist in reality only in the first case, namely when the concept is a likeness of a reality, for only then does the object signified by the concept exist in reality.[14]

We are not here concerned with the relevance of this analysis to the problem of the divine attributes, but rather with the light it throws incidentally upon the foundation of mathematics. St. Thomas classifies mathematical intentions with those of logic as having only a remote foundation in reality; their proximate basis is the intellect itself. The term St. Thomas uses to designate mathematical intentions is *abstractio mathematicorum*, which can be translated either "the abstraction of mathematicals" or "the abstraction of the mathematicians." In either case, the term does not refer to the *act* of abstracting the objects of mathematics but to the intentions themselves devised by the mathematician. St. Thomas uses *abstractio mathematicorum* simply as an example of an intention which the intellect comes upon following the way it understands reality. This is in accord with Aristotle's use of the term "abstractions" (*τὰ ἐξ ἀφαιρέσεως*) to designate mathematical objects resulting from the abstractive activity of the intellect.[15] As further examples of intentions which the intellect comes upon as a consequence of its way of

[14] "Unde sciendum, quod ipsa conceptio intellectus tripliciter se habet ad rem quae est extra animam. Aliquando enim hoc quod intellectus concipit, est similitudo rei existentis extra animam, sicut hoc quod concipitur de hoc nomine 'homo'; et talis conceptio intellectus habet fundamentum in re immediate, inquantum res ipsa, ex sua conformitate ad intellectum, facit quod intellectus sit verus, et quod nomen significans illum intellectum proprie de re dicatur. Aliquando autem hoc quod significat nomen non est similitudo rei existentis extra animam, sed est aliquid quod consequitur ex modo intelligendi rem quae est extra animam; et hujusmodi sunt intentiones quas intellectus noster adinvenit; sicut significantum hujus nominis 'genus' non est similitudo alicujus rei extra animam existentis; sed ex hoc quod intellectus intelligit animal ut in pluribus speciebus, attribuit ei intentionem generis et hujusmodi intentionis licet proximum fundamentum non sit in re, sed in intellectu, tamen remotum fundamentum est res ipsa. Unde intellectus non est falsus, qui has intentiones adinvenit. Et simile est de omnibus aliis qui consequuntur ex modo intelligendi, sicut est abstractio mathematicorum et hujusmodi. Aliquando vero id quod significatur per nomen, non habet fundamentum in re, neque proximum, neque remotum, sicut conceptio chimerae: quia neque est similitudo alicujus rei extra animam, neque consequitur ex modo intelligendi rem aliquam vere: et ideo ista conceptio est falsa. Unde patet secundum, scilicet quod ratio dicitur esse in re, inquantum significatum nominis, cui accidit esse rationem, est in re: et hoc contingit proprie quando conceptio intellectus est similitudo rei." *Ibid.*, p. 67.

[15] Aristotle, *Post. Anal.*, I, 18, 81b3. See M.-D. Philippe, 'Abstraction, Addition, Séparation dans la philosophie d'Aristote,' *Revue thomiste* XLVIII (1948) 462.

knowing, St. Thomas mentions privations and negations.[16] Intentions of this kind are secondary objects of the intellect; things themselves are its primary object.[17]

The terms used by St. Thomas to describe the intentions of mathematics in the *Scriptum* text are the same as those with which he describes an *ens rationis*. In contrast to a being of nature (*ens naturae*), he tells us, a being of reason (*ens rationis*) is properly speaking an intention which reason comes upon (*adinvenit*) in the things it considers, for example the intention of genus, species and the like, which are not found in the nature of things but follow upon the consideration of reason.

> Ens autem rationis dicitur proprie de illis intentionibus, quas ratio adinvenit in rebus consideratis; sicut intentio generis, speciei et similium, quae quidem non inveniuntur in rerum natura, sed considerationem rationis consequuntur.[18]

Similarly, in the *Scriptum* text St. Thomas describes both the objects of logic and of mathematics as intentions which the intellect hits upon (*intellectus... has intentiones adinvenit*). The term *adinvenit* has no exact equivalent in English. It does not simply mean "discovers", for we are told that these intentions are not found in reality (*non inveniuntur in rerum natura*). We do not discover them as we do forms existing in the real world. Yet they are not pure creations of the mind independent of experience. Our intellect hits upon them or devises them as a consequence of its knowledge of reality.[19] Hence both reality and the intellect have a role to play in their elaboration. In this sense they can be called elaborations of the intellect, or more simply and accurately beings of reason.

Does this mean that mathematics, like logic, has for its object beings of reason? This seems to be the implication of the *Scriptum* text. Yet St. Thomas tells us in his commentary on the *Metaphysics* that beings of reason are properly the subject of logic. In contrast to logic, the subject of philosophy (which for St. Thomas includes mathematics),[20] is real being (*ens*

[16] "Nam negatio vel privatio non est ens naturae, sed rationis, sicut dictum est." *Metaph.*, IV, lect. 1 (Turin 1935) n. 560. Cf. nn. 540, 541.

[17] "Prima enim intellecta sunt res extra animam, in quae primo intellectus intelligenda fertur. Secundą autem intellecta dicuntur intentiones consequentes modum intelligendi: hoc enim secundo intellectus intelligit in quantum reflectitur supra se ipsum, intelligens se intelligere et modum quo intelligit." *De Potentia*, VII, 9.

[18] *Metaph.*, IV, lect. 4, n. 574.

[19] St. Thomas also uses the term *adinvenit* to designate the craftman's "hitting upon" the form of the work he intends to make: "Cum enim intellectus artificis adinvenit aliquam formam artificiati, ipsa natura seu forma artificiati in se considerata, est posterior intellectu artificis." *Quodl.*, VIII, a. 1.

[20] St. Thomas classifies mathematics with metaphysics and the philosophy of nature as

naturae)[21]. Again, he makes it clear that logic alone concerns intentions.[22] Mathematics, on the other hand, has to do with quantity and its properties, for example the circle and triangle, which are forms existing in reality.[23] St. Thomas writes: "Just as the form of man exists in that special matter which is the organic body, so the form of circle or triangle exists in that special matter which is the continuum or surface or body."[24] From this it appears that, in St. Thomas' view, the objects of mathematics are real forms. Why then does he classify mathematical intentions with those of logic in his *Scriptum?*

The answer to this question is apparent if we reflect on the special character of the objects of mathematics. St. Thomas does not deny that lines and circles exist in sensible reality, but he points out that they are not of the same kind as those investigated by the mathematical sciences: *in istis sensibilibus non sunt tales lineae et tales circuli, quales scientiae mathematicae quaerunt.*[25] This is shown by the fact that mathematical entities do not always have the same properties as their real foundation. To consider but one example: mathematical circles and lines have properties which do not belong to sensible circles and lines. Euclid, for example, proves that a straight line touches a circle at only one point, but this is not true of circles and straight lines in the real world.[26]

one of the "philosophical disciplines," but he does not put it on the same level as these parts of philosophy. Arithmetic and geometry, like logic, are liberal arts, and as such they are merely preparatory to the study of philosophy. Moreover, like the other liberal arts, they involve not only knowledge but certain direct products of reason. Arithmetic, for example, involves numbering and geometry measuring. Mathematics has consequently both a cognitive and a productive side. See *Expositio super Librum Boethii De Trinitate*, V, 1; ed. Decker, pp. 164-6; ad 3m, pp. 167, 168.

[21] "Et hujusmodi, scilicet ens rationis, est proprie subjectum logicae. Hujusmodi autem intentiones intelligibiles, entibus naturae aequiparantur, eo quod omnia entia naturae sub consideratione rationis cadunt. Et ideo subjectum logicae ad omnia se extendit, de quibus ens naturae praedicatur. Unde concludit, quod subjectum logicae aequiparatur subjecto philosophiae, quod est ens naturae." *Metaph.*, IV, lect. 4, n. 574.

[22] "Sunt autem scientiae de rebus, non autem de speciebus, vel intentionibus intelligibilibus, nisi sola scientia rationalis." *De Anima*, III, lect. 8 (Turin 1936) n. 718.

[23] "Et de huiusmodi abstractis est mathematica, quae considerat quantitates et ea quae quantitates consequuntur, ut figuras et huiusmodi." *Expositio super Librum Boethii De Trinitate*, V, 3; p. 184, lines 20-22.

[24] "Sicut enim forma homini est in tali materia, quae est corpus organicum, ita forma circuli vel trianguli est in hac materia quae est continuum vel superficies vel corpus." *Metaph.*, VII, lect. 10, n. 1496.

[25] *Metaph.*, XI, lect. 1, n. 2161.

[26] "Et ideo in mathematicis oportet cognitionem secundum iudicium terminari ad imaginationem, non ad sensum, quia iudicium mathematicum superat apprehensionem sensus. Unde non est idem iudicium quandoque de linea mathematica quod est de linea sensibili, sicut in hoc quod recta linea tangit sphaeram solum secundum punctum, quod convenit rectae lineae separatae, non autem rectae lineae in materia, ut dicitur in I De anima." *Ex-*

From one point of view, therefore, the objects of mathematics are real, but from another, and perhaps more significant point of view they are devised by the intellect. They are real in the sense that they have a remote foundation in the real world. They are not, however, immediately grounded upon reality. As a consequence, the truth of mathematical judgments does not consist in their conformity with reality, nor are mathematical terms properly predicated of reality. The immediate and most significant basis of mathematical concepts is the activity of the intellect itself.

If this is true, the objects of mathematics are, for St. Thomas, beings of reason, although of a different type from those of logic. Both are the work of the intellect, but their remote foundations in reality are different. The real foundation of our logical notions is the unity in reality of the various *rationes* we conceive of it. This is exemplified by the basic logical notion of predicability. "Predication", St. Thomas writes, "is something completed by the intellect in its act of combining and dividing, having for its foundation in reality the unity of those things, one of which is said of the other. Hence the notion of predicability can be included in the notion of the intention 'genus', which is also completed by an act of the intellect."[27] The logical relation of predicability is not found in reality, but it has a remote basis there. The intellect sees that one and the same reality (for example, Socrates) is both man and animal, and consequently it knows that it can predicate animal of man in the judgment "Man is an animal." Following upon this, the intellect hits upon or devises the logical relation of predicability, which is not a real relation but one between concepts, having a remote foundation in reality.

The real foundation of mathematical notions, on the other hand, is the physical extended universe. Because the universe is extended in space and is divisible into numerically different parts, the mathematician can devise the notions of straight line, circle, whole numbers and the like. Here, as in the case of logic, what is given in experience must be "completed" by the activity of the intellect. Consequently, mathematical abstractions are not totally independent of experience, nor are they found ready made, so to speak, in reality. The importance of the *Scriptum* text lies in the fact that it shows St. Thomas' awareness of the central role in mathematics of the creative work of the intellect, by locating the immediate foundation of mathematics not in reality but in the activity of the intellect.

positio super Librum Boethii De Trinitate, VI, 2, p. 216, lines 20-26. Cf. *Metaph.*, III, lect. 7, n. 416; Aristotle, *De Anima*, I, 1, 403a12-6.

[27] "Predicatio enim est quiddam quod completur per actionem intellectus componentis et diuidentis, habens fundamentum in re ipsa unitatem eorum quorum unum de altero dicitur. Unde ratio predicabilitatis potest claudi in ratione huiusmodi intentionis que est genus, que similiter per actum intellectus completur." *De Ente et Essentia*, III; ed. Roland-Gosselin (Paris 1948) p. 29, lines 13-8.